

Moduli of (1,7)-polarized abelian surfaces via syzygies

Dedicated to the memory of Alf B. Aure

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Abstract. We prove that the moduli space $X(1,7)$ of (1,7)-polarized abelian surfaces with canonical level-structure is birational to the Fano 3-fold V_{22} of polar hexagons of the Klein quartic $\overline{X}(7)$. In particular $X(1,7)$ is rational and the birational map to \mathbb{P}^3 is defined over \mathbb{Q} . As a byproduct we obtain explicitly the equations of the (1,7)-very-ample-polarized abelian surfaces embedded in \mathbb{P}^6 .

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0. Introduction.

Moduli spaces of polarized abelian varieties is a much studied subject. The common approach to their construction is as arithmetic quotient of the Siegel upper half space. Their study involves then the beautiful subject of modular forms. In this paper we follow a different approach. We construct them as Heisenberg invariant part of a Hilbert scheme.

In the particular case of (1,7)-polarized abelian surfaces we can obtain in this way only a birational model of the moduli space, because not every polarization is very ample. However our method goes quite far: We obtain a birational parametrization of the moduli space defined over \mathbb{Q} .

Moreover we discovered a new relation between the moduli space $X(1,7)$ and the modular curve $X(7)$ of elliptic curves with a level 7 structure. The variety of sums of 6 powers of the Klein quartic $\overline{X}(7)$, ie. the variety of polar hexagons of the Klein quartic, is our model of $X(1,7)$.

The paper is organised as follows, each section devoted to one basic idea. In section 1 we review the construction of the moduli space as invariant part of the Hilbert scheme and collect some basic notations.

In section 2 we recall that a very ample line bundle of class (1,7) embeds an abelian surface projectively normal and study its syzygies. Due to $H^1(A, \mathcal{O}) \neq 0$ the minimal free resolution is longer than the codimension. However if we allow a locally free resolution, there is a rather natural self-dual resolution F . A result like this should hold quite generally for Gorenstein subvarieties of smooth manifolds, whose canonical bundle is induced, cf. [EPW], [W] for some results in this direction.

Section 3 brings in the action of the Heisenberg group. With this, the middle syzygy map boils down to a 3×2 matrix, which can be interpreted as the Hilbert-Burch matrix (cf. [E] Thm 20.15) of a twisted cubic in a certain \mathbb{P}^3 . The complex condition on F gives certain linear relations among the coefficients of the defining quadratic equations of this twisted cubic. This is enough to determine our model of the moduli space: It is a particular Fano 3-fold $V_{22} \subset \mathbb{P}^{13}$ of degree 22.

In section 4 we recall the various descriptions of a V_{22} , one of them being the variety of sums of powers of a plane quartic curve. For our V_{22} this is the Klein quartic. We finish with the birational map $\mathbb{P}^3 \dashrightarrow X(1,7)$ induced by the triple projection from a particular point of the V_{22} .

The Appendix contains some formulas of the representation theory of the Heisenberg group H_7 and $SL_2(\mathbb{Z}_7)$.

In several aspects we are not yet completely satisfied with our results here.

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1. A detailed study of the geometry of the surfaces (over boundary points) comparable to the study of Barth, Hulek and Moore [BHM] of $X(1, 5)$ via the Horrocks-Mumford bundle is missing.
2. Points on our V_{22} parametrize a family of 10-nodal Kummer quartics in \mathbb{P}^3 . Could it be that every V_{22} parametrizes some 10-nodal quartics?

However the paper had already some fruits: As observed first by Alf Aure and Kristian Ranestad, conics on the V_{22} correspond to pencils of abelian surfaces which sweep out a Calabi-Yau 3-fold Y of degree 14. In some sense the study of pairs $\{(A, Y) \mid A \subset Y\}$ of abelian surfaces contained in some Calabi-Yau is easier than studying A 's alone. This point of view turned out to be the key in the solution of Gross and Popescu [GP] of the problem posed by Gritsenko [Gr] to decide, which Siegel modular 3-folds are rational.

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Notation. Most of the notation will be introduced directly in the text. We recall here only some of it and also some notation which will be used tacitly:

A an abelian variety

\mathcal{L} an ample line bundle of type $(1, 7)$ on A

$\hat{A} = \text{Pic}^0(A)$

t_x is the automorphism of translation by $x \in A$

V^* for the dual of a vector space V

UV or $U \cdot V$ for $U \otimes V$, where U, V are vector spaces

nV for $\oplus_1^n V$

$\mathbb{G}(k, V)$ for the Grassmann variety of k dimensional subspaces of the vector space V

$\mathbb{P}(V)$ the projective space of lines in V .

We shall use the Macaulay short hand notation for numerical data of a free resolutions over the graded polynomial ring R and for its sheafified version. A table like

$$\begin{array}{ccccccc} 1 & - & - & - & - & & \\ - & 7 & 8 & - & - & & \\ - & - & 3 & 8 & 3 & & \end{array}$$

stands for a complex

$$R \leftarrow F_1 \leftarrow F_2 \leftarrow F_3 \leftarrow F_4 \leftarrow 0$$

with 5 terms $F_0 = R$ and $F_i = \oplus_{k=1}^{r_i} R(-a_{ik})$ for $i = 1, \dots, 4$ where the number of generators of F_i in a given degree are encoded by the numbers in the i^{th} column. More precisely the number in position $(i, -j)$ in the table is the number of generators of degree $i + j$ of F_i . In the example above, the image of $F_1 = 7R(-2)$ is an ideal generated by 7 quadrics, which have 8 linear syzygies and further 3 quadratic syzygies, corresponding to $F_2 = 8R(-3) \oplus 3R(-4)$; $F_3 = 8R(-5)$, $F_4 = 3R(-6)$.

Notice that the entries of a block in a syzygy matrix above corresponding to two consecutive numbers in the same line are linear, while the maps to the upper left and from the lower right corners of a square are quadratic. Examples: The syzygies of the twisted cubic in \mathbb{P}^3 have shape

$$\begin{array}{cccc} 1 & - & - & \\ - & 3 & 2 & \end{array} \quad .$$

The syzygies of a plane cubic in \mathbb{P}^3 union a point, which is not in that plane, look like

$$\begin{array}{cccc} 1 & - & - & - \\ - & 3 & 3 & 1 \\ - & 1 & 1 & - \end{array} \quad .$$

1. Review. We review some well known facts about abelian varieties following [Mum66] and [Mum70] (see also [LB]).

(1.1) Let A be an abelian variety with an ample line bundle \mathcal{L} ,

$$\phi_{\mathcal{L}} : A \rightarrow \text{Pic}^0(A), \quad x \mapsto t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$$

the associated group homomorphism, and let $K_{\mathcal{L}}$ be the kernel of $\Phi_{\mathcal{L}}$. $K_{\mathcal{L}}$ is finite, since \mathcal{L} is ample (cf. [Mum70], II.6. Application 1). Moreover, by the theorem of the cube, $\Phi_{t_y^* \mathcal{L}} = \Phi_{\mathcal{L}}$, i.e. $\Phi_{\mathcal{L}}$ depends only on $c_1(\mathcal{L})$ (the *polarization class* of \mathcal{L}).

By definition, $K_{\mathcal{L}}$ operates on $\mathbb{P}(H^0(A, \mathcal{L}))$, but not on $H^0(A, \mathcal{L})$. The pullback central extension group $G_{\mathcal{L}}$ is called the (infinite) Heisenberg group of \mathcal{L} :

$$\begin{array}{ccccccc} 1 & \rightarrow & \mathbb{C}^* & \rightarrow & G_{\mathcal{L}} & \rightarrow & K_{\mathcal{L}} \rightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 1 & \rightarrow & \mathbb{C}^* & \rightarrow & GL(H^0(A, \mathcal{L})) & \rightarrow & PGL(H^0(A, \mathcal{L})) \rightarrow 1 \end{array}$$

We will work mainly with a finite version of $G_{\mathcal{L}}$: Since $G_{\mathcal{L}}$ is a central extension of an abelian group, taking commutators induces a skew bilinear map

$$e^{\mathcal{L}} : K_{\mathcal{L}} \times K_{\mathcal{L}} \rightarrow \mathbb{C}^*, \quad e^{\mathcal{L}}(x, y) = \tilde{x} \cdot \tilde{y} \cdot \tilde{x}^{-1} \cdot \tilde{y}^{-1} \quad (\tilde{x}, \tilde{y} \in G_{\mathcal{L}} \text{ preimages of } x, y).$$

Since $K_{\mathcal{L}}$ is finite, we may replace \mathbb{C}^* by the finite group $\mu_{\mathcal{L}}$ generated by the image of $e^{\mathcal{L}}$. Moreover $|\mu_{\mathcal{L}}|$ is a divisor of the exponent of $K_{\mathcal{L}}$. The (infinite) Heisenberg group $G_{\mathcal{L}}$ is the pushout of a certain (finite) Heisenberg group $H_{\mathcal{L}}$:

$$\begin{array}{ccccccc} 1 & \rightarrow & \mu_{\mathcal{L}} & \rightarrow & H_{\mathcal{L}} & \rightarrow & K_{\mathcal{L}} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \rightarrow & \mathbb{C}^* & \rightarrow & G_{\mathcal{L}} & \rightarrow & K_{\mathcal{L}} \rightarrow 0 \end{array}.$$

(1.2) An explicit description of $H_{\mathcal{L}}$ uses more notation. In the analytic context $c_1(\mathcal{L}) \in H^2(A^{an}, \mathbb{Z}) \cong \Lambda^2 \text{Hom}(\Gamma, \mathbb{Z})$, where Γ is a lattice in \mathbb{C}^g such that $A \cong \mathbb{C}^g / \Gamma$. Hence $c_1(\mathcal{L})$ is an alternating nondegenerated 2-form on Γ with values in \mathbb{Z} , (cf. [Mum70] p. 16). In a convenient basis of Γ this alternating form can be written as a matrix:

$$c_1(\mathcal{L}) = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}$$

where Δ is a diagonal matrix (d_1, \dots, d_g) of positive integers such that $d_1 \mid d_2 \mid \dots \mid d_g$. The collection of elementary divisors $(d_1, \dots, d_g) =: \delta$ is called the type of the line bundle \mathcal{L} .

The diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \Gamma & \rightarrow & \mathbb{C}^g & \rightarrow & A \rightarrow 0 \\ & & c_1(\mathcal{L}) \downarrow & & \downarrow \cong & & \downarrow \Phi_{\mathcal{L}} \\ 0 & \rightarrow & \hat{\Gamma} & \rightarrow & \hat{\mathbb{C}}^g & \rightarrow & \hat{A} \rightarrow 0 \end{array}$$

yields an isomorphism $K_{\mathcal{L}} \cong \hat{\Gamma} / \Gamma \cong (\mathbb{Z}/d_1\mathbb{Z})^2 \oplus \dots \oplus (\mathbb{Z}/d_g\mathbb{Z})^2$.

In the algebraic setting the elementary divisors of $K_{\mathcal{L}}$ define the type: $e^{\mathcal{L}}$ is nondegenerated and produces a decomposition of $K_{\mathcal{L}}$ as a direct sum of two subgroups K_1, K_2 , where $K_2 \cong \hat{K}_1 \cong \text{Hom}(K_1, \mathbb{C}^*)$, (cf. [Mum66], p. 293). The most convenient description of $H_{\mathcal{L}}$, up to isomorphism, is as the following extension H_{δ} :

$$1 \rightarrow \mu_{d_g} \rightarrow H_{\delta} \rightarrow ((\mathbb{Z}/d_1\mathbb{Z}) \oplus \dots \oplus ((\mathbb{Z}/d_g\mathbb{Z})) \oplus (\mu_{d_1} \times \dots \times \mu_{d_g}) \rightarrow 0,$$

where $\mu_d = \text{Hom}(\mathbb{Z}/d\mathbb{Z}, \mathbb{C}^*)$ is the group of the d -th roots of unity. If we write the last term in the above exact sequence as $K(\delta) \oplus \hat{K}(\delta)$, with $\hat{K}(\delta) = \text{Hom}(K(\delta), \mathbb{C}^*)$, then the multiplication in $H_{\delta} = \mu_{d_g} \oplus K(\delta) \oplus \hat{K}(\delta)$ is defined by

$$(\alpha, x, \rho) \circ (\beta, y, \sigma) = (\alpha \cdot \beta \cdot \sigma(x), x + y, \rho \cdot \sigma).$$

On $K(\delta) \oplus \hat{K}(\delta)$ one has also naturally an alternate multiplicative form e^δ . If e_1, \dots, e_g and e_1^*, \dots, e_g^* are the canonical basis of $K(\delta)$ and $\hat{K}(\delta)$ respectively then

$$e^\delta : (K(\delta) \oplus \hat{K}(\delta)) \times (K(\delta) \oplus \hat{K}(\delta)) \rightarrow \mu_{d_g}$$

is defined by:

$$e^\delta(e_r^*, e_r) = (e^\delta(e_r, e_r^*))^{-1} = \exp(2\pi i/d_r), \text{ for all } r = 1, \dots, g \text{ and } 1 \text{ otherwise}$$

(1.3) $H^0(A, \mathcal{L})$ is an irreducible $H_{\mathcal{L}}$ -module. The argument uses that $K(\delta)$ lifts to a subgroup of $H_{\mathcal{L}}$, and that \mathcal{L} descends to $A/K(\delta)$, (cf. [Mum66], pp. 290, 297). $H^0(A, \mathcal{L})$ is the unique irreducible representation of $H_{\mathcal{L}}$, on which the center $\mu_{\mathcal{L}} \subset \mathbb{C}^*$ acts by scalar multiplication. This $(\prod_{i=1}^g d_i)$ -dimensional representation V is called the Schrödinger representation of H_δ .

(1.4) **Definition.** ([Mum66]) A **theta-structure** on the pair (A, \mathcal{L}) is any isomorphism $\alpha : H_{\mathcal{L}} \rightarrow H_\delta$ which induces the identity on the centers $\mu_{\mathcal{L}}$ and μ_{d_g} viewed as subgroups of \mathbb{C}^* . A theta-structure induces a **level-structure of canonical type** i.e. a symplectic isomorphism $\alpha' : K_{\mathcal{L}} \rightarrow K_\delta$, symplectic with respect to $e^{\mathcal{L}}$ and e^δ .

Since $\Phi_{\mathcal{L}}$ depends only on the numerical equivalence class of \mathcal{L} we can speak of a level-structure for a polarized abelian variety.

The reason to consider level-structures is the following:

(1.5) **Theorem.** (Mumford) *There exists a coarse moduli space $\mathcal{M}(\delta)$ for ample polarized abelian varieties with level-structure of type $\delta = (d_1, \dots, d_g)$.*

(1.6) If $d_1 \geq 3$ the proof of this is elementary: By Lefschetz's theorem an ample line bundle \mathcal{L} of type $\delta = (d_1, \dots, d_g)$ with $d_1 \geq 3$ is very ample. The irreducibility of the Schrödinger representation V and the level-structure on (A, \mathcal{L}) gives a canonical identification $\mathbb{P}(V) = \mathbb{P}(H^0(A, \mathcal{L}))$ by Schurs lemma. Thus every pair (A, \mathcal{L}) with level structure of type δ occurs as a point in the Hilbert scheme $Hilb(\mathbb{P}(V))$. Moreover $A \subset \mathbb{P}(V)$ is H_δ -invariant. So we can take as $\mathcal{M}(\delta)$ an open part of an irreducible component of the fixpoint set $Hilb(\mathbb{P}(V))^{H_\delta}$. To see that this has the right (in fact reduced) scheme structure we compare the tangent spaces:

$$T_{Hilb^{H_\delta}, A} = H^0(A, \mathcal{N}_A)^{H_\delta} \cong Im(H^0(A, \mathcal{N}_A) \rightarrow H^1(A, \mathcal{T}_A)).$$

Indeed, from the exact sequences:

$$\begin{aligned} 0 \rightarrow \mathcal{T}_A \rightarrow \mathcal{T} \rightarrow \mathcal{N}_A \rightarrow 0 \\ 0 \rightarrow \mathcal{O}_A \rightarrow V\mathcal{O}_A(1) \rightarrow \mathcal{T} \rightarrow 0 \end{aligned}$$

where \mathcal{T} is the restriction to A of the tangent bundle of $\mathbb{P}(V)$, one deduces the exact sequence:

$$\begin{array}{ccccccc} 0 & \rightarrow & H^0(\mathcal{T}_A) & \rightarrow & H^0(\mathcal{T}) & \rightarrow & H^0(\mathcal{N}_A) \rightarrow H^1(\mathcal{T}_A) \rightarrow \dots \\ & & \parallel & & \parallel & & \parallel \\ & & gI & & gI \oplus Z & & \frac{g(g-1)}{2}I \end{array}$$

where I is the trivial 1-representation of H_δ and Z is the complement of gI in $H^0(\mathcal{T})$. This shows that the image of $H^0(\mathcal{N}_A) = Z \oplus H^0(\mathcal{N}_A)^{H_\delta}$ in $H^1(\mathcal{T}_A)$ is $H^0(\mathcal{N}_A)^{H_\delta}$. \square

(1.7) **Remarks.** (1) Notice that the universal family over the Hilbert scheme is not necessarily a universal family in the sense of moduli, because, as subscheme, A has no distinguished origin. Indeed picking the origin appropriate we obtain that $\mathcal{O}(1)$ restricts to any desired line bundle \mathcal{L} within the polarization class $c_1(\mathcal{L})$.

(2) In general $Hilb(\mathbb{P}(V))^{H_\delta}$ has many components. For example in the elliptic curve case, only if d_1 is prime, there is a single component. For composed numbers, there are several components whose points correspond to union of d_1/n elliptic curves of degree n , for every divisor n of d_1 , (cf. [P]).

2. Syzygies.

(2.1) According to a theorem of Reider, a line bundle of type $(1, d)$, $d \geq 5$ on a general abelian surface is very ample (cf. [R] or [LB] p. 301-302). In this section we describe the syzygies of an abelian surface $A \subset \mathbb{P}^6$ embedded by a very ample line bundle of type $(1, 7)$. Although the canonical class ω_A is induced from \mathbb{P}^6 the minimal free resolution is not symmetric, since the coordinate ring is not projectively Cohen-Macaulay. However the only obstacle for this is $H^1(A, \mathcal{O})$. Using also some locally (not globally) free sheaves in the resolution we obtain a nice self-dual resolution.

(2.2) Let (A, \mathcal{L}) be an abelian surface with a very ample line bundle of type $(1, 7)$, and consider its image

$$A \hookrightarrow \mathbb{P}^6 = \mathbb{P}(V),$$

as an $H_7 = H_{(1,7)}$ -invariant subvariety.

(2.3) **Lemma.** $A \subset \mathbb{P}^6$ is not contained in a quadric.

Proof. If a quadric would contain A then $h^0(\mathbb{P}^6, \mathcal{I}_A(2)) \geq 7$, since $H^0(\mathbb{P}^6, \mathcal{O}(2))$ is a direct sum of irreducible representations of dimension 7, cf. Appendix. This is too much: By Castelnuovo's argument the 14 points $Z = A \cap \mathbb{P}^4$ for a general $\mathbb{P}^4 \subset \mathbb{P}^6$ impose at least 9 conditions on quadrics, i.e. $h^0(\mathbb{P}^4, \mathcal{I}_Z(2)) \leq 6$. On the other hand $h^0(\mathbb{P}^4, \mathcal{I}_Z(2)) = h^0(\mathbb{P}^6, \mathcal{I}_A(2)) + 2 \geq 9$ by the exact sequence

$$(2.3.1) \quad 0 \rightarrow \mathcal{I}_A \rightarrow 2\mathcal{I}_A(1) \rightarrow \mathcal{I}_A(2) \rightarrow \mathcal{I}_{Z, \mathbb{P}^4}(2) \rightarrow 0,$$

a contradiction. □

(2.4) **Corollary.** $A \subset \mathbb{P}^6$ has syzygies:

$$\begin{array}{cccccc} 1 & - & - & - & - & - \\ - & - & - & - & - & - \\ - & 21 & 49 & 42 & 14 & 2 \\ - & - & - & - & 1 & - \end{array}$$

Proof. The map $H^0(\mathbb{P}^6, \mathcal{O}(2)) \rightarrow H^0(A, \mathcal{L}^{\otimes 2})$ is injective by the Lemma, hence an isomorphism, because both spaces are 28-dimensional. So $H^1(\mathbb{P}^6, \mathcal{I}_A(2)) = 0$, i.e. $A \subset \mathbb{P}^6$ is quadratically normal. By the above: $h^0(\mathbb{P}^4, \mathcal{I}_Z(2)) = 2$. So $Z \subset \mathbb{P}^4$ has the Hilbert function

$$(1, 5, 13, 14, 14, \dots),$$

because its different function has no negative value, being the Hilbert function of an artinian ring. It follows $H^1(\mathbb{P}^4, \mathcal{I}_Z(n)) = 0$ for $n \geq 3$ and induction with the sequence (2.3.1) gives: $A \subset \mathbb{P}^6$ is projectively normal.

Moreover \mathcal{I}_A is 4-regular in the sense of Castelnuovo-Mumford and nonzero syzygy numbers can only be in the indicated range, some of whose values are clear:

$$\begin{array}{cccccc} 1 & - & - & - & - & - \\ - & - & - & - & - & - \\ - & 21 & ? & ? & ? & 2 \\ - & ? & ? & i & 1 & - \end{array}$$

Namely, the number of cubic generators of the ideal is $h^0(\mathbb{P}^6, \mathcal{I}_A(3)) = h^0(\mathbb{P}^6, \mathcal{O}(3)) - h^0(A, \mathcal{O}_A(3)) = 21$. The last 2 represents $h^1(\mathcal{O}_A)$ and the 1 comes from the facts that $\omega_A \cong \mathcal{O}_A$ and that dualizing the above resolution one obtains $\mathcal{E}xt_{\mathcal{O}_{\mathbb{P}}}^4(\mathcal{O}_A, \omega_{\mathbb{P}}) \cong \omega_A$.

The last argument gives also $i = 0$, since $\mathcal{E}xt_{\mathcal{O}_{\mathbb{P}}}^i(\mathcal{O}_A, \omega_{\mathbb{P}}) = 0$ for $i \leq 3$ and A is non-degenerate. Now all the vanishing is clear, and the nonzero values can be computed from the Hilbert function. □

The resolution has $length > codim A$, since $A \subset \mathbb{P}^6$ is not arithmetically Cohen-Macaulay. In particular it is not symmetric. However, if we allow locally free sheaves instead of only direct sums of line bundles, then there is a nice self-dual resolution:

(2.5) **Theorem.** $A \subset \mathbb{P}^6$ has a self-dual resolution of type

$$0 \leftarrow \mathcal{O}_A \leftarrow \mathcal{O} \xleftarrow{\beta} 21\mathcal{O}(-3) \xleftarrow{\alpha} 2\Omega^3 \xleftarrow{\alpha'} 21\mathcal{O}(-4) \xleftarrow{\beta'} \mathcal{O}(-7) \leftarrow 0$$

with $\alpha' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} {}^t\alpha$ and $\beta' = {}^t\beta$.

Proof. The above resolution is obtained by a kind of subtracting a piece of the Koszul sequence multiplied with $h^1(A, \mathcal{O}_A)$ from the resolution in Corollary (2.4). Let \mathcal{K} be the kernel of the map $\mathcal{O} \leftarrow 21\mathcal{O}(-3)$ in the resolution in (2.4). Then comparing the Koszul resolution of the \mathbb{C} vector space $H^1(A, \mathcal{O}_A)^* \cong \text{Ext}_S^5(S_A, S(-7))$ with the dual complex of (2.4), yields a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ & & 21\mathcal{O}(-4) & & & & 0 \\ & & \downarrow & & & & \downarrow \\ 0 & \leftarrow & 2 \cdot \Omega^3 & \leftarrow & 2 \cdot 35\mathcal{O}(-4) & \leftarrow & 2 \cdot 21\mathcal{O}(-5) & \leftarrow & 2 \cdot 7\mathcal{O}(-6) & \leftarrow & 2 \cdot \mathcal{O}(-7) & \leftarrow & 0 \\ & & \downarrow & & \downarrow \wr & & \downarrow & & \parallel & & & & \\ 0 & \leftarrow & \mathcal{K} & \leftarrow & 49\mathcal{O}(-4) & \leftarrow & 42\mathcal{O}(-5) & \leftarrow & 14\mathcal{O}(-6) \oplus \mathcal{O}(-7) & \leftarrow & 2\mathcal{O}(-7) & \leftarrow & 0 \\ & & \downarrow & & & & \downarrow & & & & & & \\ & & 0 & & & & \mathcal{O}(-7) & & & & & & \\ & & & & & & \downarrow & & & & & & \\ & & & & & & 0 & & & & & & \end{array}$$

The map $2 \cdot 21\mathcal{O}(-5) \rightarrow 42\mathcal{O}(-5)$ is surjective, and $\ker(2 \cdot 35\mathcal{O}(-4) \rightarrow 49\mathcal{O}(-4)) \cong 21\mathcal{O}(-4)$, because otherwise $A \subset \mathbb{P}^6$ would be contained in a quadric, or more then 21 cubics. A diagram chase gives the desired resolution. To see the assertions about the maps, we compare this complex with its dual:

$$\begin{array}{ccccccc} 0 & \leftarrow & \mathcal{O}_A & \leftarrow & \mathcal{O} & \xleftarrow{\beta} & 21\mathcal{O}(-3) & \xleftarrow{\alpha} & H^1(\mathcal{O}_A) \otimes \Omega^3 & \xleftarrow{\alpha'} & 21\mathcal{O}(-3) & \xleftarrow{\beta'} & \mathcal{O}(-7) & \leftarrow & 0 \\ & & \downarrow \wr \phi & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr u & & \downarrow \wr & & \downarrow \wr & & \\ 0 & \leftarrow & \omega_A & \leftarrow & \mathcal{O} & \xleftarrow{{}^t\beta'} & 21\mathcal{O}(-3) & \xleftarrow{{}^t\alpha'} & H^1(\mathcal{O}_A)^* \otimes \Omega^3 & \xleftarrow{{}^t\alpha} & 21\mathcal{O}(-3) & \xleftarrow{{}^t\beta} & \mathcal{O}(-7) & \leftarrow & 0 \end{array}$$

The isomorphism u is compatible with Serre duality. The diagrams:

$$\begin{array}{ccccccc} H^1(\mathcal{O}_A) & \cong & H^1(\mathcal{O}_A) \otimes \mathbb{C} & & H^1(\mathcal{O}_A) \otimes H^1(\mathcal{O}_A) & \longrightarrow & H^2(\mathcal{O}_A) \\ H^1(\phi) \downarrow \wr & & \downarrow \wr u & \text{and} & \downarrow id \otimes H^1(\phi) & & \downarrow H^2(\phi) \\ H^1(\omega_A) & \xrightarrow{\cong} & H^1(\mathcal{O}_A)^* \otimes \mathbb{C} & & H^1(\mathcal{O}_A) \otimes H^1(\omega_A) & \longrightarrow & H^2(\omega_A) \end{array}$$

commute. So the map in the first row of the last diagram is antisymmetric, as the one in the second row is. This gives the relations between α' , β' and α , respectively β . \square

3. Symmetry. Taking into account the symmetries of (A, \mathcal{L}) we have the following H_7 or G_7 -invariant resolutions.

(3.1) **Corollary.** *Taking the canonical H_7 -invariant embedding of A corresponding to the taken polarization and the level structure, one gets:*

$$0 \leftarrow \mathcal{I}_A \leftarrow 3V_4\mathcal{O}(-3) \leftarrow 7V_1\mathcal{O}(-4) \leftarrow 6V_2\mathcal{O}(-5) \leftarrow 2V\mathcal{O}(-6) \oplus \mathcal{O}(-7) \leftarrow 2\mathcal{O}(-7) \leftarrow 0$$

If one considers an G_7 -invariant embedding one obtains:

$$0 \leftarrow \mathcal{I}_A \leftarrow 3V_4\mathcal{O}(-3) \leftarrow (5V_1 \oplus 2V_1^\sharp)\mathcal{O}(-4) \leftarrow 6V_2^\sharp\mathcal{O}(-5) \leftarrow 2V^\sharp\mathcal{O}(-6) \oplus \mathcal{O}(-7) \leftarrow 2S\mathcal{O}(-7) \leftarrow 0$$

Proof. Everything follows using the tables from the appendix. \square

(3.2) **Theorem.** *An abelian surfaces G_7 -invariantly embedded in \mathbb{P}^6 has an G_7 -invariant resolution of the form:*

$$0 \leftarrow \mathcal{O}_A \leftarrow \mathcal{O} \xleftarrow{\beta} 3V_4\mathcal{O}(-3) \xleftarrow{\alpha} 2S\Omega^3 \xleftarrow{\alpha'} 3V_1\mathcal{O}(-4) \xleftarrow{\beta'} \mathcal{O}(-7) \leftarrow 0$$

with $\alpha' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} {}^t\alpha$ and $\beta' = {}^t\beta$.

Proof. The big diagram in the proof of theorem 2.5 reads as G_7 -modules

$$\begin{array}{ccccccc}
0 & & & & & & \\
\downarrow & & & & & & \\
3V_1\mathcal{O}(-4) & & & & & & 0 \\
\downarrow & & & & & & \downarrow \\
0 \leftarrow 2S \cdot \Omega^3 \leftarrow 2S \cdot (4V_1^\sharp \oplus V_1)\mathcal{O}(-4) \leftarrow 2S \cdot (3V_2)\mathcal{O}(-5) \leftarrow & & 2S \cdot V\mathcal{O}(-6) & \leftarrow & 2S \cdot \mathcal{O}(-7) \leftarrow 0 \\
\downarrow & & \parallel & & \downarrow & & \parallel \\
0 \leftarrow \mathcal{K} \leftarrow (5V_1 \oplus 2V_1^\sharp)\mathcal{O}(-4) \leftarrow 6V_2^\sharp\mathcal{O}(-5) \leftarrow 2V^\sharp\mathcal{O}(-6) \oplus \mathcal{O}(-7) \leftarrow 2S \cdot \mathcal{O}(-7) \leftarrow 0 \\
\downarrow & & & & \downarrow & & \\
0 & & & & \mathcal{O}(-7) & & \\
& & & & \downarrow & & \\
& & & & 0 & &
\end{array}$$

and the result is clear. \square

Next we view α as a 3×2 -matrix with entries in $\text{Hom}(S\Omega^3, V_4\mathcal{O}(-3))$. Since α defines a G_7 -morphism the entries lie in the G_7 -invariant part.

(3.3) **Proposition.**

$$\text{Hom}_{G_7}(S\Omega^3, V_4\mathcal{O}(-3)) = 4I,$$

i.e. α has entries in a 4-dimensional vector space.

Proof. $\text{Hom}(\Omega^3, \mathcal{O}(-3)) \cong \Lambda^3 V = V_1 \oplus 4V_1^\sharp$. Hence $\text{Hom}(S\Omega^3, V_4\mathcal{O}(-3)) \cong V_4 \otimes (V_1^\sharp \oplus 4V_1) = 4I \oplus S \oplus 5Z$ and $\text{Hom}_{G_7}(S\Omega^3, V_4\mathcal{O}(-3)) = 4I$. \square

(3.4) **Remark.** If \mathcal{F}_1 and \mathcal{F}_2 are two G_7 -sheaves, $\text{Hom}_{G_7}(\mathcal{F}_1, \mathcal{F}_2)$ is a N -module, because $G_7 = H_7 \rtimes \mathbb{Z}_2$ is a normal subgroup of $N \cong H_7 \rtimes SL_2(\mathbb{Z}_7)$, ι being central in $SL_2(\mathbb{Z}_7)$.

Using the character table of $SL_2(\mathbb{Z}_7)$, one sees that $\text{Hom}_{G_7}(S\Omega^3, V_4\mathcal{O}(-3)) \cong U'$, with the notation from the appendix.

For the following considerations we choose a basis u_0, \dots, u_3 of U' , so that in the decomposition into irreducible G_7 -modules of $\Lambda^3 V = V_1 \oplus (U' \otimes V_1)$, u_0, \dots, u_3 correspond to the V_1 pieces generated as a H_7 -module by $e_1 \wedge e_4 \wedge e_2 - e_6 \wedge e_3 \wedge e_5$, $e_0 \wedge e_1 \wedge e_6$, $e_0 \wedge e_2 \wedge e_5$, or $e_0 \wedge e_4 \wedge e_3$ respectively. Then, in the above decomposition of $\Lambda^3 V$, V_1 is generated as a H_7 -module by $e_1 \wedge e_4 \wedge e_2 + e_6 \wedge e_3 \wedge e_5$. More precisely, the above elements correspond to $u_0 \otimes e_0, \dots, u_3 \otimes e_0$ in $U' \otimes V_1$ and the elements $u_k \otimes e_\ell$ are obtained permuting the indices of e 's via σ .

With these notations the matrix $\alpha = (a_{ij})$ will have entries $(a_{ij}) = \sum_{k=0}^3 a_{ij}^k u_k$. We want to express more conveniently the condition $\alpha\alpha' = 0$, where $\alpha' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \alpha$.

(3.5) **Proposition.** A matrix α as above satisfies $\alpha\alpha' = 0$ iff the three quadrics in $\mathbb{P}^3 = \mathbb{P}(U)$ given by its 2×2 -minors are annihilated by each of the three operators:

$$\Delta_1 = \frac{\partial^2}{\partial u_0 \partial u_1} - \frac{1}{2} \frac{\partial^2}{\partial u_2^2}, \quad \Delta_2 = \frac{\partial^2}{\partial u_0 \partial u_2} - \frac{1}{2} \frac{\partial^2}{\partial u_3^2}, \quad \Delta_3 = \frac{\partial^2}{\partial u_0 \partial u_3} - \frac{1}{2} \frac{\partial^2}{\partial u_1^2}.$$

Proof. We use the fact that under the identifications $\text{Hom}(\mathcal{O}(-4), \Omega^3) = \Lambda^3 V$, $\text{Hom}(\Omega^3, \mathcal{O}(-3)) = \Lambda^3 V$ the composition of two maps $\mathcal{O}(-4) \rightarrow \Omega^3, \Omega^3 \rightarrow \mathcal{O}(-3)$ is given by wedge product, if we identify canonically $\wedge^6 V$ with $V^* = V_3 = H^0(\mathcal{O}(1))$.

Observe now that u_0 interpreted as an element in $\text{Hom}(V_1\mathcal{O}(-4), S\Omega^3)$ is given by the following 1×7 matrix with entries in $\Lambda^3 V$:

$$u_0 = (e_{1+k} \wedge e_{4+k} \wedge e_{2+k} - e_{6+k} \wedge e_{3+k} \wedge e_{5+k})_{k \in \mathbb{Z}_7}$$

and similarly:

$$u_1 = (e_k \wedge e_{1+k} \wedge e_{6+k})_k, \quad u_2 = (e_k \wedge e_{2+k} \wedge e_{5+k})_k, \quad u_3 = (e_k \wedge e_{4+k} \wedge e_{3+k})_k.$$

The same elements, interpreted in $\text{Hom}(S\Omega^3, V_4\mathcal{O}(-3))$, will be identified with the transpose of the above ones. Then the only compositions of two u_i 's which are not 0 are:

$$u_0 u_1 = u_1 u_0 = -u_2 u_2 = B_1 := \begin{pmatrix} 0 & x_4 & 0 & 0 & 0 & 0 & -x_3 \\ -x_4 & 0 & x_5 & 0 & 0 & 0 & 0 \\ 0 & -x_5 & 0 & x_6 & 0 & 0 & 0 \\ 0 & 0 & -x_6 & 0 & x_0 & 0 & 0 \\ 0 & 0 & 0 & -x_0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & 0 & -x_1 & 0 & x_2 \\ x_3 & 0 & 0 & 0 & 0 & -x_2 & 0 \end{pmatrix}$$

$$u_0 u_2 = u_2 u_0 = -u_3 u_3 = B_2 := \begin{pmatrix} 0 & 0 & x_1 & 0 & 0 & -x_6 & 0 \\ 0 & 0 & 0 & x_2 & 0 & 0 & -x_0 \\ -x_1 & 0 & 0 & 0 & x_3 & 0 & 0 \\ 0 & -x_2 & 0 & 0 & 0 & x_4 & 0 \\ 0 & 0 & -x_3 & 0 & 0 & 0 & x_5 \\ x_6 & 0 & 0 & -x_4 & 0 & 0 & 0 \\ 0 & x_0 & 0 & 0 & -x_5 & 0 & 0 \end{pmatrix}$$

$$u_0u_3 = u_3u_0 = -u_1u_1 = B_3 := \begin{pmatrix} 0 & 0 & 0 & -x_5 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_6 & x_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x_0 & x_4 \\ x_5 & 0 & 0 & 0 & 0 & 0 & -x_1 \\ -x_2 & x_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x_3 & x_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x_4 & x_1 & 0 & 0 & 0 \end{pmatrix}.$$

In particular $u_iu_j = u_ju_i$, i.e. the maps commute. Now, in the interpretation of $\alpha = (a_{ij})$ as a 3×2 matrix each entry is a linear combination of the u_j 's.

$$\begin{aligned} \alpha\alpha' &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{pmatrix} \\ &= \begin{pmatrix} 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{32} - a_{12}a_{31} \\ a_{21}a_{12} - a_{22}a_{11} & 0 & a_{21}a_{32} - a_{22}a_{31} \\ a_{31}a_{12} - a_{32}a_{11} & a_{31}a_{22} - a_{32}a_{21} & 0 \end{pmatrix} \end{aligned}$$

is a 3×3 matrix with each entry being a 7×7 block matrix, which is a linear combination of the three linearly independent matrices B_j above. The condition that this composition is zero says that the quadrics defined as the 2×2 -minors of the matrix α , considered as quadrics in the u_j 's, have the coefficients of u_1^2 , u_2^2 , u_3^2 respectively equal with the coefficients of u_0u_3 , u_0u_1 , u_0u_2 . \square

(3.6) Proposition. *If a matrix α with $\alpha\alpha' = 0$ comes from an exact complex, then the three quadrics given by its 2×2 -minors are linearly independent.*

Proof. Assume

$$\alpha = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

is a matrix as above, with three linear dependent 2×2 -minors. Without any loss of generality, may assume

$$\alpha = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \alpha = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \\ 0 & a_{32} \end{pmatrix}.$$

Indeed, after a linear change of the rows we may assume that the minor corresponding of the 2^{nd} and 3^{rd} row is zero. A 2×2 determinant of linear forms is zero, iff either two rows or the two columns are linearly dependent. A further base change gives α the shape above.

Case 1. If α has a zero row then the ideal contains a summand $V_4\mathcal{O}(-3)$, a contradiction.

Case 2. In the second case we first note that $a_{11} = l_0u_0 + l_1u_1 + l_2u_2 + l_3u_3$ is non-zero. Indeed, otherwise

$$0 \leftarrow I_A \leftarrow 3V_4\mathcal{O}(-3) \leftarrow S\Omega^3 \leftarrow 0$$

would be exact, and A could have codimension 2 at most.

Consider now

$$V_4\mathcal{O}(-3) \xleftarrow{a_{11}} S\Omega^3 \xleftarrow{(u_0, u_1, u_2, u_3)} 4V_1\mathcal{O}(-4).$$

The composition is the concatenation of the four 7×7 block matrices $l_1B_1 + l_2B_2 + l_3B_3 \mid l_0B_1 - l_1B_3 \mid l_0B_2 - l_2B_1 \mid l_0B_3 - l_3B_2$. Precisely 2 blocks are linearly independent, because on one hand $a_{11} \cdot a_{22} = a_{11} \cdot a_{23} = 0$, on the other hand the ideal J in (4.1) has too few syzygies to allow an subideal of type $a_{11} \cdot (a, b, c)$. So we get seven cubics with two skew symmetric 7×7 matrices of relations. Since all 4 skew symmetric matrices have rang 6 in a general point of \mathbb{P}^6 (eg. in the point $(1 : 2 : 3 : 4 : 5 : 6 : 7)$) unless they are identically zero, there is up to scalar a unique set of cubics whose relations they are: In each case the 7 principal pfaffians. These pfaffians are not proportional for two different blocks for any values (l_0, \dots, l_3)

unless the blocks themselves are proportional, as can be seen by a straight forward computation. This is the desired contradiction. \square

4. Moduli. Denote by $X(1, 7)^v$ the open set of abelian surfaces with a very ample polarization of class $(1, 7)$. For each $A \in X(1, 7)^v$ we choose a G_7 -equivariant embedding $A \hookrightarrow \mathbb{P}^6$. Its syzygy determine a 3×2 matrix $\alpha = \alpha_A$ as in Theorem 3.2. α is determined by A up to conjugation with $GL(3, \mathbb{C}) \times SL(2, \mathbb{C})$. The ideal $I = I_A \subset S = \mathbb{C}[u_0, u_1, u_2, u_3]$ of minors of α_A is uniquely determined by A . We denote by $C_A \subset \mathbb{P}(U)$ the zero loci of I_A .

(4.1) **Proposition.** $C_A \subset \mathbb{P}(U)$ is a projectively Cohen-Macaulay curve of degree 3 and arithmetic genus 0.

Proof. Denote by S the graded ring $\mathbb{C}[u_0, u_1, u_2, u_3] = S(U') = \oplus_{\ell \geq 0} S^\ell U'$. By the Hilbert-Burch Theorem (cf. [E], Thm 20.15) the complex

$$0 \longleftarrow S/I \longleftarrow S \longleftarrow \oplus_1^3 S(-2) \xleftarrow{\alpha} \oplus_1^2 S(-3) \longleftarrow 0$$

is exact unless the three quadric minors of α have a common factor. Since the quadrics are linearly independent by Proposition 3.6, the second possibility occurs only if S/I has syzygies

$$\begin{array}{cccc} 1 & - & - & - \\ - & 3 & 3 & 1 \end{array}.$$

However

$$J := ((\Delta_1, \Delta_2, \Delta_3)^\perp) = (u_1 u_2, u_2 u_3, u_3 u_1, u_1^2 + u_0 u_3, u_3^2 + u_0 u_2, u_2^2 + u_0 u_1, u_0^2)$$

has syzygies

$$\begin{array}{cccccc} 1 & - & - & - & - \\ - & 7 & 8 & - & - \\ - & - & 3 & 8 & 3 \end{array}$$

and $I \subset J$ by Proposition 3.5. So the second possibility cannot occur and the Hilbert-Burch complex is exact. \square

(4.2) **Corollary.** A is uniquely determined by C_A .

Proof. C_A determines the Hilbert-Burch matrix α up to conjugation, which in turn determines α', β' hence I_A . \square

(4.3) The Hilbert scheme $Hilb_{3t+1}(\mathbb{P}^3)$ has two components of dimension 12 and 15 (cf. [PS]):

$$Hilb_{3t+1}(\mathbb{P}^3) = H_1 \cup H_2$$

with general points of H_1, H_2 and the intersection $H_1 \cap H_2$ corresponding respectively to a twisted cubic, a plane cubic union a point or a plane nodal cubic with an embedded point at the node. For all $C \in H_1$, $h^0(\mathbb{P}^3, \mathcal{I}_C) = 3$. The morphism

$$f: \begin{array}{ccc} H_1 & \longrightarrow & \mathbb{G}(3, H^0(\mathbb{P}^3, \mathcal{O}(2))) \\ C & \longmapsto & H^0(\mathbb{P}^3, \mathcal{I}_C(2)) \end{array}$$

is birational onto its image $H \subset \mathbb{G}(3, 10)$, regular precisely on $H_1 - H_1 \cap H_2$, cf. [EPS]. All varieties $H, H_1, H_2, H_1 \cap H_2, f(H_1 \cap H_2)$ are smooth.

Consider

$$H(\Delta) := H \cap \mathbb{G}(3, J_2) \subset \mathbb{G}(3, H^0(\mathbb{P}^3, \mathcal{O}(2))).$$

Since $\mathbb{G}(3, J_2)$ does not intersect $f(H_1 \cap H_2)$ we can regard $H(\Delta)$ as a subvariety of H_1 as well. $H(\Delta)$ has dimension at least 3 in every point by dimension count.

We are grateful to Geir Ellingsrud for pointing out to us, that such varieties were studied by Mukai.

(4.4) **Theorem.** $H(\Delta)$ is a smooth prime Fano 3-fold of genus 12.

Proof. Mukai [Muk89,92] proves that $H(\delta) = H \cap \mathbb{G}(3, \delta^\perp) \subset \mathbb{G}(3, H^0(\mathbb{P}^3, \mathcal{O}(2)))$ is a smooth prime Fano 3-fold for a general net of quadrics $(\delta_1, \delta_2, \delta_3)$ in $\mathbb{P}(U)$. The proof that $\Delta_1, \Delta_2, \Delta_3$ is general in this sense, i.e. that $H(\Delta)$ is a smooth connected Fano 3-fold, is postponed until we have considered different models of $H(\Delta)$.

(4.5) Consider on $L = \mathbb{C}^7$ the net $\eta_{klein} : \Lambda^2 L \rightarrow W' = \mathbb{C}^3$ of alternating forms defined by the matrix

$$\eta_{klein} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -y_1 & y_0 \\ 0 & 0 & 0 & 0 & -y_2 & 0 & y_1 \\ 0 & 0 & 0 & -y_0 & 0 & 0 & y_2 \\ 0 & 0 & y_0 & 0 & y_1 & -y_2 & 0 \\ 0 & y_2 & 0 & -y_1 & 0 & y_0 & 0 \\ y_1 & 0 & 0 & y_2 & -y_0 & 0 & 0 \\ -y_0 & -y_1 & -y_2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathbb{G}(3, L, \eta_{klein}) = \{E \in \mathbb{G}(3, L) \mid \Lambda^2 E \subset \text{Ker}(\eta_{klein} : \Lambda^2 L \rightarrow W')\}.$$

As zero loci of a section of a homogenous bundle on the Grassmanian, $\mathbb{G}(3, L, \eta_{klein})$ is a prime Fano 3-fold of genus 12, if it is smooth of expected dimension. Smoothness follows from the criterion in section 1 of [Muk89] by computation.

(4.6) Let $F = \{f = 0\} \subset \mathbb{P}^2$ be a plane quartic. The variety

$$VSP(F, 6) = \{\{l_1, \dots, l_6\} \in \text{Hilb}_6(\mathbb{P}^2) \mid f = l_1^4 + \dots + l_6^4\}$$

of sums powers presenting f was studied by Rosanes [Ros] 1873, Scorza [Sco1,2] and more recently by Mukai [Muk89,92]. It is a prime Fano 3-fold of genus 12 for general F . Consider $f_{klein} = v_1^3 v_2 + v_2^3 v_3 + v_3^3 v_1$ the well-known equation of the modular curve $\overline{X}(7) \subset \mathbb{P}^2 = \mathbb{P}(W)$ due to Felix Klein, [K] §4.

(4.7) **Theorem.**

$$H(\Delta) \cong \mathbb{G}(3, L, \eta_{klein}) \cong VSP(\overline{X}(7), 6).$$

Proof. Every prime Fano 3-fold V_{22} of genus 12 (hence degree 22) has these 3 descriptions [Muk89,92] over an algebraically closed field of characteristic 0. That these special ones correspond to each other follows from [Schr], where the relation between the defining data and the isomorphism between the different models is explained:

η_{klein} can be identified with the Tor-multiplication

$$\Lambda^2 \text{Tor}_1^S(S/J, \mathbb{C})_2 \longrightarrow \text{Tor}_2^S(S/J, \mathbb{C})_4.$$

Note that these Tor-groups are 7 respectively 3-dimensional, cf. (4.1.1), and in fact $\text{Tor}_1^S(S/J, \mathbb{C})_2 = L$ and $\text{Tor}_2^S(S/J, \mathbb{C})_4 = W'$, because the minimal resolution of $S/J = \mathbb{C} \oplus U' \oplus W'$ over S has the form:

$$0 \leftarrow S/J \leftarrow S \leftarrow LS(-2) \leftarrow M_1 S(-3) \oplus W' S(-4) \leftarrow M_1 S(-5) \leftarrow WS(-6) \leftarrow 0.$$

On the other hand the ideal $I_{pfa\text{ff}}$ generated by the 6×6 -Pfaffians of η_{klein} gives a Gorenstein ring $A = \mathbb{C}[y_0, y_1, y_2]/I_{pfa\text{ff}}$ of codimension 3 cf. [BE]. A is artinian and the dual socle generator is f_{klein} . This completes the proof of Theorem 4.7 and 4.4. \square

(4.8) **Remarks.** (1) The discriminant of the net of quadrics δ is another quartic, which comes with a natural vanishing theta characteristic, cf. Scorza [1889,1899] and [DK]. In our case this is again the Klein quartic. For general δ this is a different quartic than the dual socle quartic, see [Schr] for more details. The fact that

$$\{\text{quartics}\} \rightarrow \{\text{quartics with an odd theta characteristic}\}$$

is birational over \mathbb{C} was discovered by Scorza. A more recent treatment is given in [DK], and with different point of view in [Schr].

(2) If we take Mukai's results for granted, then it is clear that the quartic for the sum of powers has to coincide with the Klein quartic, because this curve is uniquely determined by its symmetry group: combine [ACGH] Ex. I F-17 with [H] Ex. IV 5.7 (b), or combine [H] Ex. IV 5.7 (a) and the appendix.

(4.9) **Theorem.** *The Moduli space $X(1, 7)$ is birational to $VSP(\overline{X}(7), 6)$.*

Proof. By 4.2 we have an immersion

$$X(1, 7)^v \hookrightarrow H(\Delta) \cong VSP(\overline{X}(7), 6).$$

Since both varieties are irreducible and 3-dimensional the result follows. \square

(4.10) **Theorem.** *$X(1, 7)$ is rational with the rational map to \mathbb{P}^3 defined over \mathbb{Q} .*

Proof. It suffices to prove that $\mathbb{G}(3, L, \eta_{klein})$ is rational over \mathbb{Q} .

For a general point $p \in \mathbb{G}(3, L, \eta)$ the triple projection defined by $|H - 3p|$ defines a birational map

$$\mathbb{G}(3, L, \eta) \dashrightarrow \mathbb{P}^3,$$

(oral communication of Mukai). Its base loci consists of the 6 conics passing through p . This map is defined over \mathbb{Q} , if the point is defined over \mathbb{Q} .

However the only readily visible rational point of $\mathbb{G}(3, V, \eta_{klein})$ is the point p_e corresponding to the curve $C_e \subset \mathbb{P}^3$ defined by $u_1 u_2 = u_2 u_3 = u_3 u_1 = 0$. Interpreted as a sum of powers this corresponds to the degenerate presentation

$$(v_1 + \epsilon v_2)^4 - v_1^4 + (v_2 + \epsilon v_3)^4 - v_2^4 + (v_3 + \epsilon v_1)^4 - v_3^4 = 4\epsilon f_{klein},$$

viewed over $\mathbb{Q}[\epsilon]/(\epsilon^2)$. For this reason we call p_e the equational point.

From the explicit form of $\eta = \eta_{klein}$ we see three lines $L_1, L_2, L_3 \subset \mathbb{G}(3, L, \eta)$ passing through p_e . So $|H - 3p_e|$ has larger dimension than for a general point. We pass to a subsystem. $|H - 2L_1 - 2L_2 - 2L_3|$ has dimension 3. Its base loci consist of $L_1 \cup L_2 \cup L_3$ with each line with a 4-fold structure: The normal bundle of each line is $\mathcal{O}_L \oplus \mathcal{O}_L(-1)$. Hence $\mathcal{I}_L^2/\mathcal{I}_L^3$ has a summand \mathcal{O}_L and this lies in the base loci since $|H - 2L_1 - 2L_2 - 2L_3| \subset |H - 3p_e|$. Note that the 4-fold structure is not generically a complete intersection.

Two general hyperplanes $H_1, H_2 \in |H - 2L_1 - 2L_2 - 2L_3|$ intersect along each line in a 5-fold structure. Hence the residual curve has degree 7. It intersects each line in one point. Hence a third general hyperplane $H_3 \in |H - 2L_1 - 2L_2 - 2L_3|$ intersects the residual curve in these 3 points with multiplicity 2 and a single further point, i.e. $|H - 2L_1 - 2L_2 - 2L_3|$ defines a birational map to \mathbb{P}^3 .

Instead of giving all the details of the arguments above we prefer to describe the inverse map

$$\psi: \mathbb{P}^3 \dashrightarrow \mathbb{G}(3, V, \eta) = \mathbb{G}(3, V) \cap \mathbb{P}^{13} \subset \mathbb{P}^{34}$$

explicitly.

Consider the 3×7 matrix

$$\Psi = \begin{pmatrix} -t_0 t_3 & t_0 t_1 + t_2^2 & -t_3^2 & 0 & t_1 t_3 & -t_2 t_3 & 0 \\ t_1^2 + t_0 t_3 & -t_2^2 & -t_0 t_2 & -t_1 t_2 & 0 & t_2 t_3 & 0 \\ t_0 t_1^2 + t_1 t_2^2 + t_0^2 t_3 & t_2 t_3^2 & t_1^2 t_3 + t_0 t_2^2 & 0 & 0 & t_0 t_2 t_3 & t_1 t_2 t_3 \end{pmatrix}$$

The rational map from \mathbb{P}^3 with coordinates t_0, \dots, t_3 to the Grassmanian $\mathbb{G}(3, 7)$ defined by Ψ is for the given basis of V the desired rational parametrization ψ . ψ is bi-regular on $\mathbb{P}^3 - \{t_1 t_2 t_3 = 0\}$. \square

(4.11) **Corollary.** *The rational universal family of 3×2 matrices on $\mathbb{P}^3 \times \mathbb{P}^3$ with coordinates t, u is given by*

$$\alpha(t) = \begin{pmatrix} t_0 u_1 + t_2 u_2 & -t_2 u_0 & -t_1 u_1 \\ t_2 u_2 & -t_0 u_2 - t_3 u_3 & t_3 u_0 \\ u_1 & u_2 & u_3 \\ t_1 & t_2 & t_3 \end{pmatrix}.$$

Proof. The 3x3 minors of $\alpha(t)$ give 2 forms of bidegree (2,2) and two further forms of bidegree (3,2), (2,3) respectively. The 3 forms of degree 2 in the u 's are simply the product of the matrix of basis elements of J with ${}^t\Psi$. The form of degree 3 in the u 's is dependent unless $t_1 = t_2 = t_3 = 0$. Thus for any given point $[t_0 : t_1 : t_2 : t_3] \neq [1 : 0 : 0 : 0]$ the minimal version of the matrix $\alpha(t)$ is a 3 x 2 matrix of linear forms in the u 's, whose minors are the desired 3 quadrics. \square

Consider the vector

$$D = (x_0x_3x_4, x_0x_1x_6, x_0x_2x_5, x_2^2x_3 + x_4x_5^5, x_1^2x_5 + x_2x_6^2, x_4^2x_6 + x_1x_3^2, x_1x_2x_4 + x_3x_5x_6 - x_0^3)$$

of τ -invariant cubics in S^3V_3 .

(4.12) **Corollary.** *The rational family on $\mathbb{P}^3 \times \mathbb{P}^6$ defined by the H_7 -invariant subspace of cubics generated by the τ -invariant forms*

$$(g_1, g_2, g_3) = D \cdot {}^t\Psi$$

has as its fibres a dense subfamily of the universal family of G_7 -invariant abelian surfaces of type (1,7).

Proof. Observe that, with the notations from A5. one has:

$$J = (f_3, f_1, f_2, f_4, f_5, f_6, f_0)$$

and

$$D = (f_3e_0, f_1e_0, f_2e_0, f_4e_0, f_5e_0, f_6e_0, f_0e_0)$$

The entries of J and D correspond to each other as elements in two isomorphic irreducible $SL_2(\mathbb{Z}_7)$ -modules. \square

Appendix

A1. **The Heisenberg Group $H_7 = H_{(1,7)}$ and the extended variant $G_7 = H_7 \rtimes \mathbb{Z}_2$** We use notations similar to those from [HM] and [Man86], [Man89]

(A1.1) **The direct construction of $H_{(1,7)}$ as a subgroup of $SL_7(\mathbb{C})$**

Let $V = \text{Map}(\mathbb{Z}_7, \mathbb{C})$. On V consider the automorphisms σ, τ defined through:

$$\sigma x(j) = x(j+1)$$

$$\tau x(j) = \varepsilon^j x(j)$$

where $\varepsilon = \frac{e^{2\pi i}}{7} \in \mu_7$ and let H_7 be the group generated by σ, τ (the Heisenberg group $H_{(1,7)}$). Then H_7 is generated by matrices A_{ij} of the form:

$$A_{j\ell} = (\varepsilon^{aj+b} \delta_{j,\ell+c}) \quad , \text{ where } a, b, c \in \mathbb{Z}_7$$

and has order 343.

The Galois group Θ of $\mathbb{Q}(\varepsilon)$ over \mathbb{Q} acts on H and let θ be the generator given by $\theta(\varepsilon) = \varepsilon^3$. Then $\theta^3 = \text{complex conjugation}$. The group H is a central extension preserved by the action of Θ :

$$1 \rightarrow \mu_7 \rightarrow H \rightarrow \mathbb{Z}_7 \times \mathbb{Z}_7 \rightarrow 0$$

(we identified tacitly μ_7 and \mathbb{Z}_7 in the last nonzero term and $\sigma \mapsto (1,0)$, $\tau \mapsto (0,1)$).

The irreducible H -module V produces 5 more by the composition with the automorphisms $\theta^i \in \Theta$; denote by V_i the representation $H \xrightarrow{\theta^i} H \rightarrow \text{Aut} V$. These 6 representations are inequivalent, as one sees computing their characters, and together with the characters of $\mathbb{Z}_7 \times \mathbb{Z}_7$ are all irreducible characters of H .

Let $\Phi : \mathbb{Z}_7 \times \mathbb{Z}_7 \rightarrow H$ be the map

$$\Phi(m, n) = \varepsilon^{4mn} \sigma^m \tau^n$$

and consider

$$\omega : \mu_7 \times (\mathbb{Z}_7 \times \mathbb{Z}_7) \rightarrow H \quad \text{given by} \quad \omega(\alpha, z) = \alpha \Phi(z)$$

Then ω is a bijection and the product in H corresponds to

$$(\alpha, z) \cdot (\alpha', z') = (\alpha \alpha' B(z, z'), z + z') \quad \text{where} \quad B(m, n; m', n') = \varepsilon^{3(mn' - m'n)}$$

Let N be the normaliser of H in $SL_7(\mathbb{C})$. Then N is a central extension:

$$1 \rightarrow H \rightarrow N \xrightarrow{\alpha} SL_2(\mathbb{Z}_7) \rightarrow 1$$

where $\alpha(x)$ = the automorphism of $\mathbb{Z}_7 \times \mathbb{Z}_7$ preserving B , induced by conjugation by $x \in \text{Aut}(H)$.

Note that for $u \in SL_2(\mathbb{Z}_7)$ the action γ_u on H can be expressed:

$$\gamma_u \omega(\alpha, z) = \omega(\alpha, u(z)).$$

We have seen that any polarized abelian surface (A, L) with a polarization of type $(1, 7)$, L very ample and a fixed level structure is canonically embedded in $\mathbb{P}^6 := \text{the projective space of lines in } V$. Moreover, we may suppose that L is symmetric with respect to the origin of A , because, by a change of the origin, we can realize this situation. Then the map $x \mapsto -x$ on A extends to an automorphism of order 2 of \mathbb{P}^6 , induced by $\iota \in SL(V)$, $\iota x(j) = -x(-j)$. Therefore we consider from now on abelian surfaces in \mathbb{P}^6 invariant under the action of $G = H \rtimes \mathbb{Z}_2$.

Remark. If $\{e_j\}_j$ is the canonical basis of $V = \text{Map}(\mathbb{Z}_7, \mathbb{C})$, i.e. $e_j(\ell) = \delta_{j\ell}$, and $\{x_j\}_j$ is the dual basis of $V^* = V_3$, then the action of σ, τ on V and on $H^0(\mathbb{P}^6, \mathcal{O}(1)) = V^* = V_3$ is given by:

$$\begin{aligned} \sigma e_j &= e_{j-1} & \sigma x_j &= x_{j-1} \\ \tau e_j &= \varepsilon^j e_j & \tau x_j &= \varepsilon^{-j} x_j \end{aligned}$$

(A1.2) **Character table of G** (cf. [Man86] for $H_5 \rtimes \mathbb{Z}_2$):

$\{\alpha\}$	$C_{m,n}$	C_α	
1	1	1	I
$7\theta^i(\alpha)$	0	$\theta^i(\alpha)$	V_i
1	1	-1	S
$7\theta^i(\alpha)$	0	$-\theta^i(\alpha)$	V_i^\sharp
2	$\varepsilon^{sm+tn} + \varepsilon^{-sm-tn}$	0	$Z_{s,t}$

where:

$\{\alpha\}$ is the conjugacy class containing only the central element $\alpha \in \mu_7$,

$C_{m,n} = \{(\alpha, m, n), (\alpha, -m, -n) | \alpha \in \mu_7\}$ (with $(m, n) \neq (0, 0)$)

and $C_\alpha = \{(\alpha, m, n) | m, n \in \mathbb{Z}_7\}$.

Thus there are 7 classes $\{\alpha\}$, 24 classes $C_{m,n}$ (each with 14 elements) and 7 classes C_α (each with 49 elements). We denote by Z the sum of all 24 $Z_{s,t}$.

(A1.3) **Useful formulae**

We have the following formulae:

$$\begin{aligned} V_i \otimes V_i &= 3V_{i+2} \oplus 4V_{i+2}^\sharp & \wedge^2 V_i &= 3V_{i+2} & \wedge^6 V_i &= V_{i+3} \\ V_i \otimes V_{i+1} &= 3V_{i+4} \oplus 4V_{i+4}^\sharp & \wedge^3 V_i &= V_{i+1} \oplus 4V_{i+1}^\sharp & \wedge^7 V_i &= I \\ V_i \otimes V_{i+2} &= 3V_{i+1} \oplus 4V_{i+1}^\sharp & \wedge^4 V_i &= V_{i+4} \oplus 4V_{i+4}^\sharp \\ V_i \otimes V_{i+3} &= I \oplus Z & \wedge^5 V_i &= 3V_{i+5} \end{aligned}$$

$$\begin{aligned}
S^2V_i &= 4V_{i+2}^\sharp & S^{10}V_i &= 544V_{i+1} \oplus 600V_{i+1}^\sharp \\
S^3V_i &= 8V_{i+1} \oplus 4V_{i+1}^\sharp & S^{11}V_i &= 908V_{i+4} \oplus 852V_{i+4}^\sharp \\
S^4V_i &= 10V_{i+4} \oplus 20V_{i+4}^\sharp & S^{12}V_i &= \frac{1}{7} \left(\frac{1}{2} \binom{18}{6} - 42 \right) V_{i+5} \oplus \frac{1}{7} \left(\frac{1}{2} \binom{18}{6} + 42 \right) V_{i+5}^\sharp \\
S^5V_i &= 38V_{i+5} \oplus 28V_{i+5}^\sharp & S^{13}V_i &= \frac{1}{7} \left(\frac{1}{2} \binom{19}{6} - 42 \right) V_{i+3} \oplus \frac{1}{7} \left(\frac{1}{2} \binom{19}{6} + 42 \right) V_{i+3}^\sharp \\
S^6V_i &= 56V_{i+3} \oplus 76V_{i+3}^\sharp & S^{14}V_i &= 12 \cdot 384I \oplus 12 \cdot 374S \oplus 48 \cdot 618Z \\
S^7V_i &= 8I \oplus 28S \oplus 35Z & & \\
S^8V_i &= 197V_i \oplus 232V_i^\sharp & & \text{etc.} \\
S^9V_i &= 375V_{i+2} \oplus 340V_{i+2}^\sharp & &
\end{aligned}$$

$$\begin{aligned}
H^0(\Omega^3(3)) &= 0 & H^0(\Omega^3(8)) &= 405V_3 \oplus 420V_3^\sharp \\
H^0(\Omega^3(4)) &= \wedge^3V = V_1 \oplus 4V_1^\sharp & H^0(\Omega^3(9)) &= 880V_5 \oplus 880V_5^\sharp \\
H^0(\Omega^3(5)) &= 16V_2 \oplus 16V_2^\sharp & H^0(\Omega^3(10)) &= 1704V_4 \oplus 1728V_4^\sharp \\
H^0(\Omega^3(6)) &= 56V \oplus 64V^\sharp & & \text{etc.} \\
H^0(\Omega^3(7)) &= 24I \oplus 24S \oplus 49Z & &
\end{aligned}$$

Observing that the trace of ι on $H^0(\mathcal{O}_A(k))$ is -1 for k odd and 4 for k even, one deduces:

$$\begin{aligned}
H^0(\mathcal{O}_A(1)) &= V_3 & H^0(\mathcal{O}_A(8)) &= 30V_3 \oplus 34V_3^\sharp \\
H^0(\mathcal{O}_A(2)) &= 4V_5^\sharp & H^0(\mathcal{O}_A(9)) &= 41V_5 \oplus 40V_5^\sharp \\
H^0(\mathcal{O}_A(3)) &= 5V_4 \oplus 4V_4^\sharp & H^0(\mathcal{O}_A(10)) &= 48V_4 \oplus 52V_4^\sharp \\
H^0(\mathcal{O}_A(4)) &= 6V_1 \oplus 10V_1^\sharp & H^0(\mathcal{O}_A(11)) &= 61V_1 \oplus 60V_1^\sharp \\
H^0(\mathcal{O}_A(5)) &= 13V_2 \oplus 12V_2^\sharp & H^0(\mathcal{O}_A(12)) &= 70V_2 \oplus 74V_2^\sharp \\
H^0(\mathcal{O}_A(6)) &= 16V \oplus 20V^\sharp & H^0(\mathcal{O}_A(13)) &= 85V \oplus 84V^\sharp \\
H^0(\mathcal{O}_A(7)) &= 3I \oplus 4S \oplus 7Z & H^0(\mathcal{O}_A(14)) &= 16I \oplus 12S \oplus 28Z \\
& & & \text{etc.}
\end{aligned}$$

A2 The Normaliser N of H in $SL_7(\mathbb{C})$. Via the map $\alpha : N \rightarrow SL_2(\mathbb{Z}_7)$ we get, entirely like in [HM], that N is a semidirect product $H \rtimes SL_2(\mathbb{Z}_7)$. Then, in fact $N \subset SL_7(\mathbb{Q}(\varepsilon))$. Thus Θ acts on N and all V_i are also N -modules. One shows, like in [HM], that $V_i \otimes V_i^* \cong I \oplus Z$, for all i , where Z is the space of trace 0, and as a $\mathbb{Z}_7 \times \mathbb{Z}_7$ -module is the sum of all 48 nontrivial modules. In fact, as a N/μ_7 -module it is irreducible and its character takes values in \mathbb{Q} .

A3 The Table of Characters of the Group $SL_2(\mathbb{Z}_7)$ and their Multiplication. First of all we make some notations:

$$\begin{aligned}
\alpha &= i\sqrt{7} & \lambda_1 &= \varepsilon^{1^2} - \varepsilon^{-1^2} = \varepsilon - \varepsilon^6 & \eta_1 &= \varepsilon + \varepsilon^6 \\
\alpha^+ &= \frac{1}{2}(1 + \alpha) & \lambda_2 &= \varepsilon^{2^2} - \varepsilon^{-2^2} = \varepsilon^4 - \varepsilon^3 & \eta_2 &= \varepsilon^4 + \varepsilon^3 \\
\alpha^- &= \frac{1}{2}(1 - \alpha) & \lambda_3 &= \varepsilon^{3^2} - \varepsilon^{-3^3} = \varepsilon^2 - \varepsilon^5 & \eta_3 &= \varepsilon^2 + \varepsilon^5
\end{aligned}$$

Then we have the following equalities, useful in computations:

$$\begin{aligned}
\varepsilon + \varepsilon^2 + \varepsilon^4 &= -\alpha^- & \lambda_1 + \lambda_2 + \lambda_3 &= \alpha & \eta_1 + \eta_2 + \eta_3 &= -1 \\
\varepsilon^3 + \varepsilon^5 + \varepsilon^6 &= -\alpha^+ & \lambda_1 \lambda_2 \lambda_3 &= \alpha & \eta_1 \eta_2 \eta_3 &= 1
\end{aligned}$$

and

$$\begin{aligned}\lambda_1^2 &= \eta_3 - 2 & \lambda_1 \lambda_2 &= \eta_3 - \eta_2 & \alpha \eta_1 &= \lambda_1 - 2\lambda_2 \\ \lambda_2^2 &= \eta_1 - 2 & \lambda_2 \lambda_3 &= \eta_1 - \eta_3 & \alpha \eta_2 &= \lambda_2 - 2\lambda_3 \\ \lambda_3^2 &= \eta_2 - 2 & \lambda_3 \lambda_1 &= \eta_2 - \eta_1 & \alpha \eta_3 &= \lambda_3 - 2\lambda_1\end{aligned}$$

The general shape of the elements in N is:

$$A_{jk} = \pm \frac{1}{\sqrt{7}} \varepsilon^{aj^2+bjk+ck^2+dj+ek+f} \quad (a, b, \dots, f \in \mathbb{Z}_7, b \neq 0)$$

$$A_{jk} = \pm \varepsilon^{aj^2+bj+c} \delta_{j,dk+e} \quad (a, b, \dots, e \in \mathbb{Z}_7, d \neq 0)$$

where the signs are chosen to have $\det(A_{jk}) = 1$.

For the convenience of the computations, it is useful to identify some elements in $N = H \cdot SL_2(\mathbb{Z}_7)$ and their images in $SL_2(\mathbb{Z}_7)$:

$$\mu x(j) = x(2j) \quad (\text{resp. } \mu e_j = e_{j/2}) \quad \text{corresponding to } \bar{\mu} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \in SL_2(\mathbb{Z}_7)$$

$$\nu x(j) = \varepsilon^{j^2} x(j) \quad (\text{resp. } \nu e_j = \varepsilon^{j^2} e_j) \quad \text{corresponding to } \bar{\nu} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \in SL_2(\mathbb{Z}_7)$$

$$\delta x(j) = \frac{i}{\sqrt{7}} \sum_k \varepsilon^{kj} x(k) \quad (\text{resp. } \delta e_j = \frac{i}{\sqrt{7}} \sum_k \varepsilon^{kj} e_k) \quad \text{corresponding to } \bar{\delta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z}_7)$$

Observe that $\delta^2 = \iota$ and that the elements in $SL_2(\mathbb{Z}_7)$ are given according to:

$$\begin{aligned}\mu \sigma \mu^{-1} &= \sigma^2 & \iota \sigma \iota &= \sigma^{-1} & \nu \sigma \nu^{-1} &= \varepsilon^{4 \cdot 1 \cdot 2} \sigma \tau^2 & \delta \sigma \delta^{-1} &= \tau \\ \mu \tau \mu^{-1} &= \tau^4 & \iota \tau \iota &= \tau^{-1} & \nu \tau \nu^{-1} &= \tau & \delta \tau \delta^{-1} &= \sigma^{-1}\end{aligned}$$

Character Table of $SL_2(\mathbb{Z}_7)$

	id	ι	μ	$\iota\mu$	ν	ν^3	$\iota\nu^3$	$\iota\nu$	δ		
	id	$-id$	$\bar{\mu}$	$\bar{\iota\mu}$	$\bar{\nu}$	$\bar{\nu}^3$	$\bar{\iota\nu}^3$	$\bar{\iota\nu}$	$\bar{\delta}$	$\begin{pmatrix} 22 \\ 52 \end{pmatrix}$	$\begin{pmatrix} 52 \\ 55 \end{pmatrix}$
	1	1	56	56	24	24	24	24	42	42	42
I	1	1	1	1	1	1	1	1	1	1	1
M_1	8	-8	-1	1	1	1	-1	-1	0	0	0
M_2	8	8	-1	-1	1	1	1	1	0	0	0
L	7	7	1	1	0	0	0	0	-1	-1	-1
U	4	-4	1	-1	α^-	α^+	$-\alpha^+$	$-\alpha^-$	0	0	0
$U' = U^*$	4	-4	1	-1	α^+	α^-	$-\alpha^-$	$-\alpha^+$	0	0	0
T_1	6	-6	0	0	-1	-1	1	1	0	$\sqrt{2}$	$-\sqrt{2}$
T_2	6	-6	0	0	-1	-1	1	1	0	$-\sqrt{2}$	$\sqrt{2}$
T	6	6	0	0	-1	-1	-1	-1	2	0	0
W	3	3	0	0	$-\alpha^+$	$-\alpha^-$	$-\alpha^-$	$-\alpha^+$	-1	1	1
$W' = W^*$	3	3	0	0	$-\alpha^-$	$-\alpha^+$	$-\alpha^+$	$-\alpha^-$	-1	1	1

We indicate here also the multiplication table of the characters of $SL_2(\mathbb{Z}_7)$:

$$\begin{aligned}
M_1 \otimes M_1 &= I \oplus 3M_2 \oplus 3L \oplus 2T \oplus W \oplus W' \\
M_1 \otimes M_2 &= 3M_1 \oplus 2U \oplus 2U' \oplus 2T_1 \oplus 2T_2 \\
M_1 \otimes L &= 3M_1 \oplus U \oplus U' \oplus 2T_1 \oplus 2T_2 \\
M_1 \otimes U &= 2M_2 \oplus L \oplus T \oplus W \\
M_1 \otimes U' &= 2M_2 \oplus L \oplus T \oplus W' \\
M_1 \otimes T_1 &= 2M_2 \oplus 2L \oplus 2T \oplus W \oplus W' \\
M_1 \otimes T_2 &= 2M_2 \oplus 2L \oplus 2T \oplus W \oplus W' \\
M_1 \otimes T &= 2M_1 \oplus U \oplus U' \oplus 2T_1 \oplus 2T_2 \\
M_1 \otimes W &= M_1 \oplus U \oplus T_1 \oplus T_2 \\
M_1 \otimes W' &= M_1 \oplus U' \oplus T_1 \oplus T_2 \\
M_2 \otimes M_2 &= I \oplus 3M_2 \oplus 3L \oplus 2T \oplus W \oplus W' \\
M_2 \otimes L &= 3M_2 \oplus 2L \oplus 2T \oplus W \oplus W' \\
M_2 \otimes U &= 2M_1 \oplus U \oplus T_1 \oplus T_2 \\
M_2 \otimes U' &= 2M_1 \oplus U' \oplus T_1 \oplus T_2 \\
M_2 \otimes T_1 &= 2M_1 \oplus U \oplus U' \oplus 2T_1 \oplus 2T_2 \\
M_2 \otimes T_2 &= 2M_1 \oplus U \oplus U' \oplus 2T_1 \oplus 2T_2 \\
M_2 \otimes T &= 2M_2 \oplus 2L \oplus 2T \oplus W \oplus W' \\
M_2 \otimes W &= M_2 \oplus L \oplus T \oplus W \\
M_2 \otimes W' &= M_2 \oplus L \oplus T \oplus W'
\end{aligned}$$

$$\begin{aligned}
L \otimes L &= I \oplus 2M_2 \oplus 2L \oplus 2T \oplus W \oplus W' \\
L \otimes U &= M_1 \oplus U \oplus U' \oplus T_1 \oplus T_2 \\
L \otimes U' &= M_1 \oplus U \oplus U' \oplus T_1 \oplus T_2 \\
L \otimes T_1 &= 2M_1 \oplus U \oplus U' \oplus T_1 \oplus 2T_2 \\
L \otimes T_2 &= 2M_1 \oplus U \oplus U' \oplus 2T_1 \oplus T_2 \\
L \otimes T &= 2M_2 \oplus 2L \oplus T \oplus W \oplus W' \\
L \otimes W &= M_2 \oplus L \oplus T \\
L \otimes W' &= M_2 \oplus L \oplus T
\end{aligned}$$

$$\begin{aligned}
U \otimes U &= L \oplus T \oplus W \\
U \otimes U' &= I \oplus M_2 \oplus L & U' \otimes U' &= L \oplus T \oplus W' \\
U \otimes T_1 &= M_2 \oplus L \oplus T \oplus W' & U' \otimes T_1 &= M_2 \oplus L \oplus T \oplus W \\
U \otimes T_2 &= M_2 \oplus L \oplus T \oplus W' & U' \otimes T_2 &= M_2 \oplus L \oplus T \oplus W \\
U \otimes T &= M_1 \oplus U' \oplus T_1 \oplus T_2 & U' \otimes T &= M_1 \oplus U \oplus T_1 \oplus T_2 \\
U \otimes W &= T_1 \oplus T_2 & U' \otimes W &= M_1 \oplus U \\
U \otimes W' &= M_1 \oplus U' & U' \otimes W' &= T_1 \oplus T_2
\end{aligned}$$

$$\begin{aligned}
T_1 \otimes T_1 &= I \oplus 2M_2 \oplus L \oplus T \oplus W \oplus W' \\
T_1 \otimes T_2 &= 2M_2 \oplus 2L \oplus T & T_2 \otimes T_2 &= I \oplus 2M_2 \oplus L \oplus T \oplus W \oplus W' \\
T_1 \otimes T &= 2M_1 \oplus U \oplus U' \oplus T_1 \oplus T_2 & T_2 \otimes T &= 2M_1 \oplus U \oplus U' \oplus T_1 \oplus T_2 \\
T_1 \otimes W &= M_1 \oplus U' \oplus T_1 & T_2 \otimes W &= M_1 \oplus U' \oplus T_2 \\
T_1 \otimes W' &= M_1 \oplus U \oplus T_1 & T_2 \otimes W' &= M_1 \oplus U \oplus T_2
\end{aligned}$$

$$\begin{aligned}
T \otimes T &= I \oplus 2M_2 \oplus L \oplus 2T & W \otimes W &= T \oplus W' \\
T \otimes W &= M_2 \oplus L \oplus W' & W \otimes W' &= I \oplus M_2 \\
T \otimes W' &= M_2 \oplus L \oplus W & W' \otimes W' &= T \oplus W
\end{aligned}$$

A4 Multiplications, exterior, symmetric powers of N – representations, some $H^0(\Omega^3(j))$ ' s.

$$\begin{aligned}
V_{2j} \otimes V_{2j} &= (U' \oplus W') \otimes V_{2j+2} & V_{2j} \otimes V_{2j+1} &= (U \oplus W) \otimes V_{2j+4} \\
V_{2j+1} \otimes V_{2j+1} &= (U \oplus W) \otimes V_{2j+3} & V_{2j+1} \otimes V_{2j+2} &= (U' \oplus W') \otimes V_{2j+5}
\end{aligned}$$

$$\begin{aligned}
V_{2j} \otimes V_{2j+2} &= (U \oplus W) \otimes V_{2j+1} & V_j \otimes V_{j+3} &= I \oplus Z \\
V_{2j+1} \otimes V_{2j+3} &= (U' \oplus W') \otimes V_{2j+2}
\end{aligned}$$

$$\begin{aligned}
\Lambda^2 V_{2j} &= W' \otimes V_{2j+2} & \Lambda^3 V_{2j} &= (I \oplus U') \otimes V_{2j+1} & \Lambda^4 V_{2j} &= (I \oplus U) \otimes V_{2j+4} \\
\Lambda^5 V_{2j} &= W \otimes V_{2j+5}
\end{aligned}$$

$$\begin{aligned}
\Lambda^2 V_{2j+1} &= W \otimes V_{2j+3} & \Lambda^3 V_{2j+1} &= (I \oplus U) \otimes V_{2j+2} & \Lambda^4 V_{2j+1} &= (I \oplus U') \otimes V_{2j+5} \\
\Lambda^5 V_{2j+1} &= W' \otimes V_{2j+6}
\end{aligned}$$

$$\begin{aligned}
\Lambda^6 V_j &= V_{j+3} & \Lambda^7 V_j &= I
\end{aligned}$$

$$\begin{aligned}
S^2 V_{2j} &= U' \otimes V_{2j+2} \\
S^3 V_{2j} &= (I \oplus L \oplus U) \otimes V_{2j+1} \\
S^4 V_{2j} &= (L \oplus W \oplus U \oplus U' \oplus T_1 \oplus T_2) \otimes V_{2j+4} \\
S^5 V_{2j} &= (I \oplus M_1 \oplus M_2 \oplus 2L \oplus U \oplus U' \oplus T_1 \oplus T_2 \oplus 2T \oplus W') \otimes V_{2j+5} \\
&\text{etc.}
\end{aligned}$$

$$\begin{aligned}
H^0(\Omega^3(3)) &= 0 \\
H^0(\Omega^3(4)) &= (I \oplus U') \otimes V_1 \\
H^0(\Omega^3(5)) &= (L \oplus U' \oplus W' \oplus T_1 \oplus T_2 \oplus T) \otimes V_2 \\
H^0(\Omega^3(6)) &= (M_1 \oplus M_2 \oplus 3L \oplus 2U \oplus 2W \oplus W' \oplus 4T_1 \oplus 4T_2 \oplus 3T) \otimes V \\
H^0(\Omega^3(7)) &= (I \oplus 2L \oplus U \oplus 2U' \oplus W' \oplus T_1 \oplus T_2 \oplus T)(I \oplus Z) \oplus Z \\
&\text{etc.}
\end{aligned}$$

A5 Concrete Decompositions of Certain N or $SL_2(\mathbb{Z}_7)$ Representations.

Consider now the decomposition of V into eigenspaces of ι :

$$V = V^+ \oplus V^- \quad \text{where:}$$

$$V^+ = \text{span} \{e_{12} - e_{-12}, e_{22} - e_{-22}, e_{32} - e_{-32}\} = \text{span} \{e_1 - e_6, e_4 - e_3, e_2 - e_5\}$$

$$V^- = \text{span} \{2e_0, e_1 + e_6, e_4 + e_3, e_2 + e_5\}$$

Restricting μ, ν, δ to V^+ and V^- respectively, one gets:

$$\begin{aligned}
\mu^+ &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \nu^+ = \text{diag}(\varepsilon, \varepsilon^2, \varepsilon^4), \quad \delta^+ = \frac{i}{\sqrt{7}} \begin{pmatrix} \varepsilon - \varepsilon^6 & \varepsilon^4 - \varepsilon^3 & \varepsilon^2 - \varepsilon^5 \\ \varepsilon^4 - \varepsilon^3 & \varepsilon^2 - \varepsilon^5 & \varepsilon - \varepsilon^6 \\ \varepsilon^2 - \varepsilon^5 & \varepsilon - \varepsilon^6 & \varepsilon^4 - \varepsilon^3 \end{pmatrix} \\
\mu^- &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \nu^- = \text{diag}(1, \varepsilon, \varepsilon^2, \varepsilon^4), \quad \delta^- = \frac{i}{\sqrt{7}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & \varepsilon + \varepsilon^6 & \varepsilon^4 + \varepsilon^3 & \varepsilon^2 + \varepsilon^5 \\ 2 & \varepsilon^4 + \varepsilon^3 & \varepsilon^2 + \varepsilon^5 & \varepsilon + \varepsilon^6 \\ 2 & \varepsilon^2 + \varepsilon^5 & \varepsilon + \varepsilon^6 & \varepsilon^4 + \varepsilon^3 \end{pmatrix}
\end{aligned}$$

From the character table of $SL_2(\mathbb{Z}_7)$ one sees that, as a $SL_2(\mathbb{Z}_7)$ -module, $V = W' \oplus U'$ and from the above computations one gets concrete realizations of W', U' , namely $W' = V^+$ and $U' = V^-$.

$$\begin{aligned}
S^2 W &= T & S^2 W' &= T \\
S^3 W &= L \oplus W' & S^3 W' &= L \oplus W \\
S^4 W &= I \oplus M_2 \oplus T & S^4 W' &= I \oplus M_2 \oplus T
\end{aligned}$$

If we denote by

$$v_1 = e_1 - e_6 \quad v_1 = e_4 - e_3 \quad v_1 = e_2 - e_5$$

the chosen basis of W' , then the only $SL_2(\mathbb{Z}_7)$ -invariant quartic is the Klein quartic:

$$f_{\text{klein}} = v_1^3 v_2 + v_2^3 v_3 + v_3^3 v_1 \quad .$$

$$S^2 U' = L \oplus W' \quad .$$

We choose as basis for $L \subset S^2 U'$ the following elements:

$$f_0 = u_0^2, \quad f_1 = u_2 u_3, \quad f_2 = u_3 u_1, \quad f_3 = u_1 u_2, \quad f_4 = u_0 u_3 + u_1^2, \quad f_5 = u_0 u_1 + u_2^2, \quad f_6 = u_0 u_2 + u_3^2.$$

and for W' the elements

$$v_3 = u_0 u_3 - u_1^2, \quad v_2 = u_0 u_1 - u_2^2, \quad v_1 = u_0 u_2 - u_3^2.$$

Then in the decomposition

$$S^3V_3 = (I \oplus U' \oplus L)V_4$$

the elements corresponding to f_je_0 are given by:

$$\begin{array}{lll} f_0e_0 = x_1x_2x_4 + x_3x_5x_6 - x_0^3 & f_1e_0 = x_0x_1x_6 & f_4e_0 = x_2^2x_3 + x_5^2x_4 \\ f_2e_0 = x_0x_2x_5 & f_5e_0 = x_1^2x_5 + x_6^2x_2 & \\ f_3e_0 = x_0x_3x_4 & f_6e_0 = x_4^2x_6 + x_3^2x_1 & \end{array}$$

From here one obtains all f_je_k via cyclic permutation, in other words via the action of σ .

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